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PARAFREE LIE ALGEBRAS WITH CERTAIN PROPERTIES

Z. Velioglu[1], N. Ekici[2]

[1]Department of Mathematics, Harran University, Sanliurfa,Turkey
[2]Department of Mathematics, Cukurova University, Adana, Turkey
e-mail: zehrav@harran.edu.tr, nekici@cu.edu.tr

Abstract. It has been studied on parafree groups since 1967 [1]. Because of the close relationship between groups and Lie algebras, we study some properties of parafree Lie algebras that are analogous to those of parafree groups. We prove that the union of the free Lie algebras of rank two is parafree and we employ this result to construct some parafree Lie algebras with certain properties.

Keywords: Parafree Lie algebras, free Lie algebras, residually nilpotent, directed system.

AMS Subject Classification: 17B99, 17B01.

1. Introduction

In [1,2,3,4] Baumslag has introduced the notion of parafree groups and has obtained some results about parafree groups. Many questions about parafree groups have remain unanswered. Baumslag has taken his result [5,6] on one relator groups to formulate several of these questions for one-relator parafree groups. Because of the close relationship between groups and Lie algebras, one would expect that parafree Lie algebras enjoy properties that are analogous to those of parafree groups. We have taken this opportunity to obtain some results about parafree Lie algebras.

Parafree Lie algebras firstly arise in the works of Baur [7]. In 1980 Baur following on his results about parafree Lie algebras, he has given an example of a parafree Lie algebra which is not free [8]. In order to state his result, he has used some results of [9]. They have answered certain questions and obtained basic results which provide a solid understanding of the structure of parafree Lie algebras. The aim of this work is to construct parafree Lie algebras of rank two which have certain properties. We carry the formal arguments used in [2] over to parafree Lie algebras. More exactly, the following theorem is proved.

Theorem 1.1. Let F be a free Lie algebra freely generated by the set $\{a,b\}$. Then there exists a parafree Lie algebra P of rank two such that

i) $P/V_k(P) = F/V_k(F)$, for every $k \geq 1$,

ii) P is not free.

2. Notations and Definitions

Let L be a Lie algebra over a field k. The lower central series

$$L = \gamma_1 (L) \supseteq \gamma_2 (L) \supseteq .... \supseteq \gamma_n (L) \supseteq ...$$

is defined inductively by $\gamma_2 (L)=[L, L]$, $\gamma_{n+1} (L) =[\gamma_n (L) , L]$, $n\geq 1$.
If n is the smallest integer satisfying $\gamma_n (L) = 0$, then L is called nilpotent of degree n. Let $n_1, n_2,.. ,n_k$ be a sequence of positive integers with $n_i \geq 1$ for $i=1,2,. . . ,k$. We define polycentral series of L, relative to this sequence, inductively by

$$L_{n_1,n_2,\dots,n_i} = \gamma_{n_i}\left(\gamma_{n_{i-1}}\left(\cdots\left(\gamma_{n_1}(L)\right)\cdots\right)\right)$$

for $i\leq k$. In case $n_1 = n_2 =. . . =n_i = 2$ we write $L_{n_1,n_2,\dots,n_i} = \delta^i(L)$ and call it the i-th term of the derived series of L. If m is the smallest integer satisfying $\delta^m(L) = \{0\}$ then L is called solvable of degree m.

A Lie algebra is said to be hopfian if it is not isomorphic to any of its proper quotients. A Lie algebra L is hopfian if and only if every surjective endomorphism of L is an automorphism.

Definition 2.1. A Lie algebra L is called residually nilpotent if

$$\bigcap_{n=1}^{\infty} \gamma_n(L) = \{0\},$$

equivalently, given any non-trivial element $u\in L$, there exists an ideal J of L such that $u\notin J$ with L/J nilpotent. We define here the notion of parafree Lie algebras.

We associate with the lower central series of L its lower central sequence:

$$L/\gamma_2 (L) , L/\gamma_3 (L) , ...$$

We say that two Lie algebras L and H have the same lower central sequence if, $L/\gamma_n (L) \cong H/\gamma_n (H)$ for every $n \geq 1$.

Definition 2.2. The Lie algebra L is called parafree over a set X if,

i: L is residually nilpotent, and

ii: L has the same lower central sequence as a free Lie algebra generated by the set X.

The cardinality of X is called the rank of L.

Let G be a Lie algebra. If there exists a parafree Lie algebra P such that $G\cong P/\delta^i (P)$ then G is called a parafree solvable Lie algebra.

One of the crucial definition in this work is that the direct limit of Lie algebras.

Definition 2.3. Let I be a set with a partial order $\leq$. Then I is called a directed set if for any elements $i,j \in I$, there exists an element $k\in I$ such that $i\leq k$ and $j\leq k$.

Definition 2.4. Let $(I, \leq)$ be a directed set, and let $\{A_i\}_{i\in I}$ be a collection of Lie algebras indexed by I and $\varphi_{ij} : A_i \rightarrow A_j$ be a homomorphism for all $i,j \in I$ such that $i\leq j$ with the following properties:

i: φ_{ii} is the identity of A_i, and

ii: $\varphi_{ik}= \varphi_{jk}\circ \varphi_{ij}$ for all $i,j,k \in I$ such that $i\leq j\leq k$.

Then the pair $\left(\{A_i\}_{i\in I},\{\varphi_{ij}\}_{i<j}\right)$ is called a directed system over I.

Definition 2.5. Let I be a directed set and let $\left(\{A_i\}_{i\in I},\{\varphi_{ij}\}_{i<j}\right)$ be a directed system of Lie algebras over I. The direct limit $\lim\limits_{\rightarrow} A_i$ is a Lie algebra L such

that it is unique up to isomorphism and satisfies the following universal mapping property:
i: For all i,j ∈I and i≤j there are mappings φ_i: $A_i \to L$ such that $\varphi_i = \varphi_j \circ \varphi_{ij}$
ii: If there is a Lie algebra C together with maps $\pi_i : A_i \to C$ such that $\pi_i = \pi_j \circ \varphi_{ij}$, for each i≤j, then there exists a unique Lie algebra homomorphism $\pi: L \to C$ such that $\pi_i = \pi \circ \varphi_i$.

To prove our main theorem we give an alternative definition of the direct limit:

Let I be a directed set and let $\left(\{A_i\}_{i\in I}, \{\varphi_{ij}\}_{i<j}\right)$ be a directed system of Lie algebras. The direct limit of this system is defined as the disjoint union of the A_i's modulo a certain equivalence relation ' ∼' : Denote the set of equivalence classes by $\bigcup_{n=1}^{\infty} A_n/\sim$.

Here, if a_i A_i and $a_j \in A_j$, $a_i \sim a_j$ if there is some k ∈I, k≥ j, i, such that $\varphi_{ik}(a_i) = \varphi_{jk}(a_j)$. Clearly ∼ is an equivalence relation. We will write $\overline{a_i}$ for the equivalence class for an element $a_i \in A_i$. The set of equivalence classes is a Lie algebra with the operation defined by

$$[\overline{a_i}, \overline{a_j}] = \overline{[\varphi_{ik}(a_i), \varphi_{jk}(a_j)]}.$$

This Lie algebra has the same mapping property as does the direct limit. Hence

$$\lim_{\rightarrow} A_i = \bigcup_i A_i/\sim \cdots\cdots\cdots \qquad (1)$$

The proof of (1) is the same as in the group case (see [10]). A routine exercise involving universal mapping properties shows that the direct limit of a Lie algebra, if it exists, is unique up to isomorphism. Direct limits of abelian Lie algebras is always exist.

Example 2.1. Let A be an abelian Lie algebra and let $\{A_i\}_{i\in I}$ be the set of finitely generated subalgebras of A. Then, by ordering I by i≤j if $A_i \subseteq A_j$, the set I is a directed set, since for any pair i,j, the algebra $A_i + A_j$ is both finitely generated and contains A_i and A_j. If we let $\varphi_{ij} : A_i \to A_j$ be the inclusion map whenever i≤j, we have a directed system $\left(\{A_i\}_{i\in I}, \{\varphi_{ij}\}_{i<j}\right)$. Thus the direct limit $\lim_{\rightarrow} A_i$ exists and $\lim_{\rightarrow} A_i = A$.

3. Conclusion

Our major concern in this work is the existence of parafree Lie algebras with certain properties.

Theorem 3.1. For $i \geq 1$, let F_i be the free Lie algebra generated by the free generators a_i and b_i. Then $P = \bigcup_{i\in I} F_i$ is parafree.

Proof. Let I be a directed set and $F_\alpha = \langle a_\alpha, b_\alpha \rangle$ be the free Lie algebra generated by a_α and b_α for $\alpha \in I$. Consider the homomorphism $\theta_{\alpha\beta}$ from F_α into F_β defined by

$$\theta_{\alpha\beta}: a_\alpha \to a_\beta + u_\beta,$$
$$b_\alpha \to b_\beta,$$

where $u_\beta \in \gamma_2(F_\beta)$, $\alpha, \beta \in I, \alpha < \beta$. It is well known that if $u_\beta \neq 0$, then $\theta_{\alpha\beta}(F_\alpha) = H_\beta = (a_\beta + u_\beta, b_\beta)$ is a proper subalgebra of F_β.

We choose $u_\beta \neq 0$, $if\ \alpha < \beta$ and $u_\beta = 0$, $if\ \alpha = \beta$

Hence $\theta_{\alpha\alpha} = Id_{F_\alpha}$ and for all $\alpha < \beta < \gamma$, $\theta_{\beta\gamma} \circ \theta_{\alpha\beta} = \theta_{\alpha\gamma}$. This provide us

$$\left(\{\mathrm{F}_\alpha\}_{\alpha\in I}, \{\theta_{\alpha\beta}\}_{\alpha<\beta}\right)$$

is a directed system. Let P be the direct limit of this system. Thus, for $\alpha,\beta \in I$ and $\alpha<\beta$, there is a homomorphism $\theta_\alpha : F_\alpha \to P$ such that $\theta_\beta \circ \theta_{\alpha\beta} = \theta_\alpha$.

Now we consider the equivalence relation $'\sim'$ on $\bigcup_{n=1}^{\infty} F_\alpha$, which is defined in the definition of the direct limit. A short calculation shows that the set of the equivalence classes $\bigcup_{n=1}^{\infty} F_\alpha/_\sim$ is equal to $\bigcup_{n=1}^{\infty} F_\alpha$. Therefore by (1) we obtain $\lim\limits_{\rightarrow} F_\alpha = \bigcup_{n=1}^{\infty} F_\alpha$. Hence P maybe viewed as the union of its subalgebras F_α. So

$$\mathrm{P} = \bigcup_{n=1}^{\infty} F_\alpha. \cdots\cdots \qquad (2)$$

In order to prove the parafreeness of P, we need the equality

$$\gamma_n(P) = \bigcup_{i\epsilon I} \gamma_n(F_i). \cdots\cdots\cdots \qquad (3)$$

We prove this equality by recalling that P is the set of the equivalence classes, i.e. $P = \lim\limits_{\rightarrow} F_\alpha = \bigcup_{n\epsilon I} {}^{F_\alpha}/_\sim$. Then (3) follows from the definition of the Lie operation on P and the definition of the n-th lower central term of P. By straightforward computation we obtain

$$\bigcap_{n=1}^{\infty} \gamma_n(P) = \bigcup_{i\epsilon I} \left(\bigcap_{n=1}^{\infty} \gamma_n(F_i)\right).$$

Since the free Lie algebra of finite rank are residually nilpotent then

$$\bigcap_{n=1}^{\infty} \gamma_n(F_i) = 0,$$

for all $i \in I$. Hence $\bigcap_{n=1}^{\infty} \gamma_n(P) = 0$. Therefore P is residually nilpotent. We are left with the proof that P has the same lower central sequence as a free Lie algebra. Choose $0 \neq u_i \in \gamma_2(F_i)$ and consider the subalgebra H_i of F_i generated by the set $W_i = \{a_i + u_i, b_i\}$. W_i is independent modulo $\gamma_2(F_i)$. This shows that W_i generates F_i modulo $\gamma_2(F_i)$. Now consider $F_i/\gamma_2(F_i)$:

$$F_i/\gamma_2(F_i) \cong (F_i/\gamma_n(F_i))/(\gamma_2(F_i)/\gamma_n(F_i))$$
$$\cong (F_i/\gamma_n(F_i))/((\gamma_2(F_i) + \gamma_n(F_i))/\gamma_n(F_i))$$
$$= (F_i/\gamma_n(F_i))/\gamma_2(F_i/\gamma_n(F_i)),$$

where $n \geq 2$. Since W_i generates F_i modulo $\gamma_2(F_i)$ then we get that W_i generates the nilpotent Lie algebra $F_i/\gamma_n(F_i)$ modulo its derived algebra. It is well known that if a Lie algebra N is nilpotent, then a set Y generates N if and only if Y generates N modulo $\gamma_2(N)$. Hence W_i generates F_i modulo $\gamma_n(F_i)$. Thus we have

$$F_i = H_i + \gamma_n (F_i). \tag{4}$$

By the second isomorphism theorem we get

$$F_i/\gamma_n (F_i) = (H_i + \gamma_n (F_i)) /\gamma_n (F_i) \cong H_i/(H_i \cap \gamma_n(F_i)).$$

Clearly $\gamma_n(H_i) \subseteq H_i \cap \gamma_n(F_i)$, so there is a natural homomorphism from $H_i/\gamma_n(H_i)$ onto $H_i/H_i \cap \gamma_n (F_i)$ for n=2,3,... Consider the following diagram which holds for n=2,3,...

$$H_i/\gamma_n(H_i) \to H_i/(H_i \cap \gamma_n(F_i)) \cong F_i/\gamma_n(F_i)$$

The diagram shows a homomorphism from $H_i/\gamma_n(H_i)$ onto a isomorphic copy of itself. Since finitely generated free nilpotent Lie algebras are hopfian then the kernel of this homomorphism is trivial. Hence

$$H_i/\gamma_n(H_i) \cong F_i/\gamma_n(F_i) \cong H_i/H_i \cap \gamma_n(F_i).$$

Therefore

$$\gamma_n(H_i) = H_i \cap \gamma_n(F_i) \dots \dots \dots \tag{5}$$

Now suppose that $a \in P$. Then for some $i \in I, a \in F_i$. By virtue of (4) this implies

$$P \subseteq H_i + \gamma_n(F_i) \subseteq H_i + \gamma_n(P).$$

Hence

$$P = H_i + \gamma_n (P) \dots \dots \dots \tag{6}$$

Therefore invoking (5) and (6), we have

$$P/\gamma_n (P) = (H_i + \gamma_n (P)) /\gamma_n (P) \cong H_i/H_i \cap \gamma_n(P) \cong H_i/\gamma_n(H_i)$$

for every n. So P is parafree. Let $n_1, n_2 \dots, n_k$ be any positive integers. We will write $V_k (L)$ for the polycentral term $L_{n_1, n_2 \dots, n_k}$.

Theorem 3.2. Let F be a free Lie algebra freely generated by the set $\{a, b\}$. Then there exists a parafree Lie algebra P of rank two such that

i: $P/V_k(P) \cong F/V_k(F)$, for every $k \geq 1$,

ii: P is not free.

Proof. (i) Let I, F_α and homomorphism $\theta_{\alpha\beta}$: $F_\alpha \to F_\beta$ be as in the proof of the Theorem 3.1. Now define $f_{\alpha\beta}: V_k(F_\alpha) \to V_k(F_\beta)$ as restriction of $\theta_{\alpha\beta}$ to $V_k(F_\alpha)$. Clearly $f_{\alpha\alpha} = Id_{V_k(F_\alpha)}$ and $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$, where $\gamma > \beta > \alpha$.

$$\left\{V_k(F_\alpha)_{\alpha \in I}, \{f_{\alpha\beta}\}_{\alpha<\beta}\right\}$$

is a directed system. Let P be the direct limit of the free Lie algebras F_i, i=1,2,…. So $P = \bigcup_{i \in I} F_i$. Of course $V_k(P) = \bigcup_{i \in I} V_k(F_i)$.

Clearly the direct limit of the system $\left\{V_k(F_\alpha)_{\alpha \in I}, \{f_{\alpha\beta}\}_{\alpha<\beta}\right\}$ is $\bigcup_{i \in I} V_k(F_i)$. i.e. $\varinjlim V_k(F_i) = \bigcup_{i \in I} V_k(F_i)$. Hence by (2) we obtain

$$\varinjlim V_k(F_i) = V_k(P).$$

Let F be the free Lie algebra on a and b. Now consider the homomorphism $\varphi_\alpha: F_\alpha \to {}^{F}/_{V_k(F)}$ defined as

$$\varphi_\alpha : a_\alpha \to a + V_k(F)$$
$$b_\alpha \to b + V_k(F)$$

It is clear that $\varphi_\alpha = \varphi_\beta \circ \theta_{\alpha\beta}$. By the universal property of the direct limit there is a unique homomorphism

$$\varphi : P \to {}^{F}\!/_{V_k(F)}$$

with $\varphi_\alpha = \varphi \circ \theta_\alpha$. Now we compute the kernel of φ. Clearly the kernel of φ_α is $V_k(F_\alpha)$. If $x \in Ker\varphi$, then $\varphi(x) = V_k(F)$.
Since $P = \bigcup_{i \in I} F_i$ then $x \in F_\alpha$ for any $\alpha \geq 1$. Using (6) leads $x \in V_k(F_\alpha)$. Therefore

$$\varphi_\alpha(x) = V_k(F) \ldots\ldots\ldots \qquad (7)$$

Thus $x \in Ker\varphi_\alpha$. This shows that

$$Ker\varphi = \bigcup_{\alpha=1}^{\infty} Ker\varphi_\alpha = \bigcup_{\alpha=1}^{\infty} V_k(F_\alpha) = V_k(P).$$

Hence,

$${}^{P}\!/_{V_k(P)} \cong {}^{F}\!/_{V_k(F)}.$$

ii) If we take k≠1 and $n_1 = 2$ in (i) we get $V_1(P) = \gamma_2(P)$ and ${}^{P}\!/_{\gamma_2(P)} \cong {}^{F}\!/_{\gamma_2(F)}$. Hence we observe that P is parafree of rank two. Since $P = \bigcup_{n \in I} F_n$ is not finitely generated but ${}^{P}\!/_{\gamma_2(P)}$ is finitely generated then P is not free.

Corollary 3.1. There exists a parafree Lie algebra P of rank 2 which satisfies the following properties.

i: P is the union of free Lie algebras of rank two,

ii: P/$\delta^i(P)$ ≅F/$\delta^i(F)$,

iii: P is not free.

Proof. (i) and (iii) are immediate consequence of the Theorem 3.2. To prove (ii), we put $V_k(F) = P_{2,2,\ldots 2}$ (Here, there are k times 2) in theorem 3.2. Using the pact $P_{\underbrace{2,2,\ldots,2}} = \delta^i(P)$ yields the result.

We carry Theorem 3.2 over to the following theorem.

Theorem 3.3. There exists a parafree solvable Lie algebra P of rank 2 such that

i:P/$V_k(P)$ ≅F/$V_k(F)$, where F is a free solvable Lie algebra of rank two,

ii: P is a union of free solvable Lie algebras of rank two,

iii:P is not a free solvable Lie algebra.

The proof is straightforward and is omitted.

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Парасвободные алгебры Ли с определенными свойствами

З.Велиоглы[1], Н. Экиджи[2]

[1] Кафедра математики Харран университет, Шанлыурфа, Турция
[2] Кафедра математики Университет Чукурова, Адана, Турция
e-mail: zehrav@harran.edu.tr, nekici@cu.edu.tr

РЕЗЮМЕ

Исследования парасвободных групп велись еще с 1967 [1]. Из-за тесной связи между группами и алгебрами Ли, мы изучаем некоторые свойства параморфных алгебр Ли, которые походят на те из парафитных групп. Мы доказываем, что союз свободных алгебр Ли разряда два парасвободен, и мы используем этот результат чтобы построить некоторые парасвободные алгебры Ли с определенными свойствами.

Ключевые слова: Параморфные алгебры Ли, свободные алгебры Ли, остаточно нильпотентная, направленная система.